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Univalency of certain analytic functions

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Abstract. The object of the present paper is to derive some sufficient conditions for univalency of certain analytic functions in the open unit disk. The univalency of certain integral operators of analytic functions is also considered.

1 Introduction

Let A denote the class of functions $f(z)$ normalized by $f(0)=f'(0)-1=0$ that are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. We denote by S the subclass of A consisting of functions which are univalent in U.
In this paper, we shall use the following result due to Pommerenke ([4], [5]).

Lemma 1. *Let r_0 be a real number, $0<r_0\leq 1$, $U_{r_0}=\{z\in\mathbb{C}:|z|<r_0\}$ and let $f(z,t)=a_1(t)z+a_2(t)z^2+\cdots$, $a_1\neq 0$,be regular in U_{r_0}, for all $t\geqq 0$ and locally absolutely continuous in $I=[0,\infty)$, locally uniformly with respect to U_{r_0}. Suppose that for almost all $t\in I$, $f(z,t)$ satisfies the equation*

$$z\frac{\partial f(z,t)}{\partial z}=p(z,t)\frac{\partial f(z,t)}{\partial t}, \tag{1}$$

for $z\in U_{r_0}$, where $p(z,t)$ is regular in U_{r_0} and $\mathrm{Re}p(z,t)>0$*, for all $t\in I$. If $|a_1(t)|\to\infty$ for $t\to\infty$ and if $f(z,t)/a_1(t)$ forms a normaly family in U_{r_0}, for all $t\in I$, $f(z,t)$ is a regular and univalent extension to the whole disk U.*

Ozaki and Nunokawa ([2]) have shown

Lemma 2. *Let $f(z)\in A$ satisfy*

$$\left|\frac{z^2f'(z)}{f(z)^2}-1\right|<1 \qquad (z\in U), \tag{2}$$

then $f(z)$ is univalent in U,

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The next lemma was given by Becker ([1]).

Lemma 3. *If $f(z)$ belonging to A satisfies*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3}$$

for all $z \in U$, then $f(z)$ is univalent in U.

Furthermore, for the integral operator of analytic functions, we need the following lemma due to Pascu ([3]).

Lemma 4. *Let α be a complex number with* $\mathrm{Re}(\alpha) > 0$ *and $f(z)$ be in the class A. It $f(z)$ saisfies*

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \qquad (z \in U), \tag{4}$$

then the integral operator

$$F_\alpha(z) = \left\{\alpha \int_0^z u^{\alpha-1} f'(u)du\right\}^{\frac{1}{\alpha}} \tag{5}$$

is in the class S.

2 Sufficient conditions for univalency

Our first theorem for sufficient conditions for univalency is contained in

Theorem 1. *Let $f(z) \in A$, c be a complex number with $|c| \leqq 1$, $c \neq -1$. And let $s=a+ib$, $\sigma=\alpha+i\beta$ be complex numbers with $a > 0$, $\alpha > 0$. If*

$$|c+1-K| < |K| \tag{6}$$

and

$$\left|ce^{-(s+\sigma)t} + (1-e^{-(s+\sigma)t})e^{-st}\frac{zf''(e^{-st}z)}{f'(e^{-st}z)} + 1 - K\right| < |K| \tag{7}$$

for all $z \in U$, $t \in I = [0,\infty)$, where $K = (s+\sigma)/2\alpha$, then the function $f(z)$ is in the clas S.

Proof. Because the function $f(z)$ is regular in U, it results that the function $L(z,t)$ defined by

$$L(z,t) = f(e^{-st}z) + \frac{1}{1+c}(e^{\sigma t} - e^{-st})zf'(e^{-st}z) \tag{8}$$

is regular in U, for all $t \geq 0$ and, hence, $L(z,t) = a_1(t)z + \cdots$,where

$$a_1(t) = \frac{c}{1+c}e^{-st} + \frac{1}{1+c}e^{\sigma t}. \tag{9}$$

Let's us prove that $a_1(t) \neq 0$ for all $t \geq 0$. We observe that if $a_1(t) = 0$ then from (9) it results that $c = -e^{(s+\sigma)t}$ and $|c| > 1$. Because from hypothesis that $|c| \leq 1$ and $c \neq -1$, it results that $a_1(t) \neq 0$, for all $t \geq 0$ and

$$\lim_{t\to\infty} |a_1(t)| = \infty. \tag{10}$$

It is easy to prove that, if $r_0 \in (0,1)$, then $L(z,t)/a_1(t)$ is a normal family in U_{r_0}. Since $f(ze^{-st})$ is regular in U, we have

$$\frac{\partial L(z,t)}{\partial t} = \frac{1}{1+c}(-cse^{-st} + \sigma e^{\sigma t})zf'(e^{-st}z) - se^{-st}(e^{\sigma t} - e^{-st})z^2 f''(e^{-st}z). \tag{11}$$

Because the functions $f(z)$, $f'(z)$, $f''(z)$ are regular in U, it results that, for all $r_0 \in (0,1]$, there exist numbers P, Q, R which depend upon r_0 such that

$$|f(z)| \leq P, \qquad |f'(z)| \leq Q, \qquad |f''(z)| \leq R \tag{12}$$

for all $z \in U_{r_0}$.
Let $T > 0$ be a fixed real number. Then, from (11) and (12), we have that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| \leq \frac{1}{1+c}(cs + \sigma e^{\sigma T})Q + s(e^{\sigma T} + 1)R \tag{13}$$

for all $z \in U_{r_0}$ and $t \in [0,T]$.
It follows that a constant $M > 0$ exists satisfying

$$\left|\frac{\partial L(z,t)}{\partial t}\right| \leq M \tag{14}$$

for all $z \in U_{r_0}$ and $t \geq 0$. We see, from (10), that the function $L(z,t)$ is locally absolutely continuous in I, and locally uniform with respect to U. Let us define the function $p(z,t)$ by

$$p(z,t) = \frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}, \tag{15}$$

Then, in order to prove that the function $p(z,t)$ is regular and has a positive real part in U, it is sufficient to show that the function $w(z,t)$ given by

$$w(z,t)=\frac{p(z,t)-1}{p(z,t)+1} \tag{16}$$

is regular and

$$|w(z,t)|<1 \tag{17}$$

for all $z\in U$ and $t\in I$. A simple calculation yields

$$w(z,t)=\frac{(1+s)(ce^{-(s+\sigma)t}+(1-e^{-(s+\sigma)t})\frac{e^{-st}zf''(e^{-st}z)}{f'(e^{-st}z)})}{(1-s)(ce^{-(s+\sigma)t}+(1-e^{-(s+\sigma)t})\frac{e^{-st}zf''(e^{-st}z)}{f'(e^{-st}z)})}. \tag{18}$$

If $H(z,t)$ is the function defined by

$$H(z,t)=ce^{-(s+\sigma)t}+(1-e^{-(s+\sigma)t})\frac{e^{-st}zf''(e^{-st}z)}{f'(e^{-st}z)} \tag{19}$$

and

$$X=\mathrm{Re}H(z,t),\qquad Y=\mathrm{Im}H(z,t), \tag{20}$$

then from (18) we obtain

$$w(z,t)=\frac{(1+s)(X+iY)+(1-\sigma)}{(1-s)(X+iY)+(1-\sigma)}. \tag{21}$$

The inequality (17) is equivalent to the inequality

$$|w(z,t)|^2=\frac{((1+a)X-bY+1-\alpha)^2+((1+a)Y+bX-\beta)^2}{((1-a)X-bY+1+\alpha)^2+((1-a)Y-bX+\beta)^2}<1, \tag{22}$$

if

$$X^2+Y^2-\frac{\alpha-a}{a}X-\frac{\beta+b}{a}Y-\frac{\alpha}{a}<0 \tag{23}$$

or

$$\left|X+iY+1-\frac{s+\sigma}{2a}\right|<\frac{|s+\sigma|}{2a}. \tag{24}$$

We conclude that the inequality (24) has the form

$$\left| ce^{-(s+\sigma)t} + (1 - e^{-(s+\sigma)t})e^{-st}\frac{zf''(e^{-st}z)}{f'(e^{-st}z)} + 1 - K \right| < |K| \tag{25}$$

for all $z \in U, t > 0$, which is identical to the inequality (7).
For $t = 0$, the inequality (25) has the form

$$|c + 1 - K| < |K| \tag{26}$$

and is identical with the inequality (6). Because the inequality (7) holds true for all $z \in U$ and $t \geq 0$ from the hypothesis, we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.If the function $w(z,t)$ has a singular point $z_0 \in U$, then z_0 ia a pole for the function $w(z,t)$, and hence $\lim_{z \to z_0} |w(z,t)| = \infty$, which is in contradiction with the inequality $|w(z,t)| < 1$. It follows that the function $w(z,t)$ is regular in U for all $t \geq 0$.
Finally, by means of Lemma 1, we prove that $L(z,t)$ is univalent in U for all $t \geq 0$, and for $t = 0$ $L(z,0) \equiv f(z)$, which shows $f(z)$ is in the class S.

□

3 Integral operators

For integral operators of analytic functions, we derive

Theorem 2. *Let $f(z) \in A$ satisfy the inequality (2) of Lemma 2, and let α be a complex number with $|\alpha| \leq \frac{1}{3}$. If $f(z)$ satisfies $|f(z)| \leq 1$ for all $z \in U$, then the integral operator*

$$F_\alpha(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\alpha du \tag{27}$$

belongs to S.

Proof. Note that $F_\alpha(z)$ is analytic in U and satisfies

$$F'_\alpha(z) = \left(\frac{f(z)}{z}\right)^\alpha,$$

$$F''_\alpha(z) = \alpha \left(\frac{f(z)}{z}\right)^{\alpha-1} \frac{zf'(z) - f(z)}{z^2},$$

and

$$(1 - |z|^2) \left|\frac{zF''_\alpha(z)}{F'_\alpha(z)}\right| = |\alpha| \left|\frac{zf'(z)}{f(z)} - 1\right| (1 - |z|^2). \tag{28}$$

Using (28) and Schwarz lemma, we see that

$$(1 - |z|^2) \left|\frac{zF''_\alpha(z)}{F'_\alpha(z)}\right| \leq |\alpha| \left|\frac{zf'(z)}{f(z)}\right| (1 - |z|^2) + |\alpha| (1 - |z|^2)$$

$$= |\alpha| \left|\frac{z^2 f'(z)}{f(z)^2}\right| \frac{1}{|z|} |f(z)| (1-|z|^2) + |\alpha| (1-|z|^2)$$

$$\leq |\alpha| \left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| (1-|z|^2) + 2|\alpha| (1-|z|^2). \tag{29}$$

Since $f(z)$ satisfies (2), it follows from (29) that

$$(1-|z|^2) \left|\frac{zF''_\alpha(z)}{F'_\alpha(z)}\right| \leq 3|\alpha| (1-|z|^2) \leq 3|\alpha| \leq 1 \tag{30}$$

for all $z \in U$, and for $|\alpha| \leq \frac{1}{3}$. Therefore, applying Lemma 3, we complete the proof of the theorem.

□

Finally we prove

Theorem 3. *Let $g(z) \in A$ satisfy the inequality (2), and let α be a complex number with* $\mathrm{Re}(\alpha) \geq 3$. *If $g(z)$ satisfies $|g(z)| \leq 1$ for all $z \in U$, then the integral operator*

$$G_\alpha(z) = \left\{\alpha \int_0^z u^{\alpha-1} \left(\frac{g(u)}{u}\right) du\right\}^{\frac{1}{\alpha}} \tag{31}$$

belongs to S.

Proof. Let us consider the function $f(z)$ given by

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right) du. \tag{32}$$

Then the function $f(z)$ is analytic in U and satisfies

$$f'(z) = \frac{g(z)}{z}, \qquad f''(z) = \frac{zg'(z) - g(z)}{z^2},$$

and

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \left|\frac{zf''(z)}{f'(z)}\right| = \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \left|\frac{zg'(z)}{g(z)} - 1\right|. \tag{33}$$

This implies that

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \left|\frac{zg'(z)}{g(z)}\right| + \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \tag{34}$$

for all $z \in U$. Thus we have

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \frac{1}{|z|} \left|\frac{z^2 g'(z)}{g(z)^2}\right| |g(z)| + \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \tag{35}$$

for all $z \in U$. Since Schwarz lemma leads (35) to

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left(\left|\frac{z^2g'(z)}{g(z)^2}-1\right|+2\right) \leq \frac{3}{\mathrm{Re}(\alpha)} \leq 1 \tag{36}$$

for all $z \in U$ and for $\mathrm{Re}(\alpha) \geq 3$.Consequently, noting that $f'(z) = \frac{g(z)}{z}$, and applying Lemma 4, we complete the proof.

□

References

[1] J.Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichchte Funcktionen*,J. Reine Angew. Math. **255**(1972),23-43.

[2] S.Ozaki and M.Nunokawa, *The Schwarzian derivative and univalent functions*,Proc. Amer. Math. Soc. **32(2)**(1972),393-394.

[3] N.N.Pascu, *On a univalence criterion II*,Itinerant seminar on functional equations approximation and convexity, Cluj-Napoca,Preprint No.6(1985),153-154.

[4] Ch.Pommerenke, *Uber die Subordination analytischer Funcktionen*,J. Reine Angew. Math. **218**(1965),159-173.

[5] Ch.Pommerenke, *Univalent Functions*,Vandenhoeck and Ruprecht, Gottingen(1975).

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